Prisms and prismatic cohomology.
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Joint w/ Scholze.
Original ~~title~~ title: canonical deformations of de Rham cohomology.

Everything in talk is p-complete for a fixed prime p.

I) Goals: Describe a uniform, purely site theoretic framework for capturing canonical deformations of dR cohomology.

a) Crystalline cohomology. k perfect of char $p>0$, R/k smooth.
With Frobenius $\circlearrowright$ $R\Gamma_{crys}(R/W)$ is a canonical def. of $R\Gamma_{dR}(R/k)$ to W.
Canonical and $R\Gamma_{crys}(R/W)\otimes_W k \simeq R\Gamma_{dR}(R/k)$.

Key tool: pd thickenings. Ex: $D_{\mathbb{Z}_p[x]}((x))$ pd envelope of (x).
$$\simeq \mathbb{Z}_p[x, \tfrac{x^2}{2!}, \tfrac{x^3}{3!}, \dots].$$

One goal is to make this more finitistic.

b) A_{inf}-cohomology (BMS).
$$\mathcal{O} := \mathbb{Z}_p^{cycl} \simeq \mathbb{Z}_p[\mu_{p^\infty}]^\wedge \xleftarrow{\simeq} \mathbb{Z}_p[q^{1/p^\infty}]/[p]_q$$
$$[p]_q = \frac{q^p-1}{q-1}.$$
q formal variable.
$q \longmapsto \zeta_p$.

Now, $A_{inf} := \mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,[p]_q)} \simeq W(\mathcal{O}/(p)^{perf})$.
This works for any perfectoid ring containing μ_{p^∞}.

$A_{inf} \longrightarrow \mathcal{O}$

plays the role of $W(k) \longrightarrow k$.

Take $R/\mathbb{Z}_p^{cycl}$ formally smooth. $A\Omega_R$ $\circlearrowright \phi$ Frobenius an A_{inf}-def. of $R\Gamma_{dR}(R/\mathbb{Z}_p^{cycl})$. Related to étale cohomology of generic fiber.

Key tools: almost mathematics.

c) q-dR cohomology.

$A = \mathbb{Z}_p[[q-1]]$ (takes the place of $A_{\inf}$). "de-perfectified"

R a smooth $\mathbb{Z}_p$-algebra. Actually, back up.

$R = \mathbb{Z}_p[x]^\wedge_p$.

$$q\Omega^*_{R,\square} := R[[q-1]] \xrightarrow{\nabla_q} R[[q-1]]$$

This also comes up in local coord. description of $A\Omega_R$.

Some kind of framing.

Derivation is q-deformed.

$$f(x) \longmapsto \frac{f(qx) - f(x)}{qx - x}\, dx.$$

$$x^i \longmapsto \frac{q^i - 1}{q-1} x^{i-1} dx$$

If $q \to 1$ this goes to $x^{i-1}dx$.

Observations: 1) Reduce mod $q-1$ and you get de Rham $\Omega^\bullet_{R/\mathbb{Z}_p}$.

2) $H^i(q\Omega^\bullet_{R/\square} / [p]_q) \simeq \Omega^i_{R/\mathbb{Z}_p} \otimes \mathbb{Z}_p[\zeta_p]$.

$q \mapsto \zeta_p$

q

Cartier iso!

Conj (Scholze). This is ind. of the choice of coordinates.

Get an A-valued cohomology theory on smooth $\mathbb{Z}_p$-algebras.

d) Breuil-Kisin modules. Should be a canonical def.

II) δ - rings.

Theory of δ-rings due to Joyal, Buium.

<u>Def</u>. A δ-ring is a ring A with $\delta : A \to A$ s.t.

a) $\delta(x+y) = \delta(x) + \delta(y) - \dfrac{(x+y)^p - x^p - y^p}{p}$

b) $\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\,\delta(x)\delta(y)$.

c) $\delta(1) = 0$.

<u>Rmk</u>. (1) If A is p-torsion free, then a δ-structure on A is the same as a lift of Frobenius on A/p to A.

$$\delta \longmapsto \phi(x) = x^p + p\delta(x).$$

Producing ϕ always works. Producing δ from ϕ requires p-torsion free.

(2) δ is a p-derivation. Lowers p-adic vanishing.

$$\delta(p^n) = p^{n-1} \cdot (\text{unit}).$$

This follows from

$$\delta(p^n) = \frac{p^n - p^{np}}{p} = p^{n-1}(1 - p^{np-n}).$$

δ-structure = p-derivation.

Consequence: no δ-rings over $\mathbb{Z}/(p^n)$. Indeed, no p^n can be zero by iterating δ. Suppose $p^n = 0$, $p^{n-1} \neq 0$. Then, $0 = \delta(p^n) = p^{n-1} \cdot \text{unit}$. Impossible.

(3) δ-rings has all limits, colimits. Preserved by forgetful functor to rings.

$$\{\text{rings}\} \underset{W = \text{right adjoint}}{\overset{L = \text{left adj.}}{\rightleftarrows}} \{\delta\text{-rings}\}$$

$L(\mathbb{Z}_p[x])$ = free δ-ring on one gen.

$$\mathbb{Z}_p\{x\} \cong \mathbb{Z}_p[x, \delta x, \delta^2 x, \delta^3 x, \dots]$$

W is the Witt vector functor.

(4) Say a δ-ring A is perfect if ϕ is an iso.

Lemma. If A is a δ-ring, $f \in A$ s.t. $pf = 0$, then $\phi(f) = 0$.

Cor. Perfect rings are p-torsion-free.

proof of lemma.

$$0 = \delta(pf) = p^p \delta(f) + f^p \delta(p) + p\,\delta(f)\,\delta(p)$$
$$= p^p \delta(f) + \phi(f)\,\underset{\text{Unit.}}{\underset{\uparrow}{\delta(p)}}$$

So, STS $0 = p^p \delta(f) = p^{p-1}\, p\,\delta(f) = p^{p-1}(\phi(f) - f^p)$ (since ϕ is a ring homomorphism.)
$= 0 - 0$
$= 0$

Cor. $\{\text{perfect, } p\text{-adically complete } \delta\text{-rings}\} \underset{\text{Reduce mod } p.}{\overset{\omega}{\simeq}} \{\text{perfect } \mathbb{F}_p\text{-algebras}\}$.

In particular, Frobenius lift is uniquely determined.

III) Prisms.

Def. 1) An element ξ in a δ-ring A is primitive if $\delta(\xi)$ is a unit. Ex. $\xi = p$.

2) A prism is a pair $(A, (\xi))$ where A is a δ-ring, [redacted] ξ is a primitive element, A is p-tf, δ-tf, and $F: A/(p) \to A/(p)$ is flat.
Here, (ξ) is the ideal.

This condition can be removed by deriving.

Remark. 1) Better to use locally, replacing ξ with an effective Cartier divisor.

2) Again can get rid of flatness by deriving.

Crys. cohomology.

Exs. 1) $A = \mathbb{Z}_p$, $\xi = p$. $(A, (p))$ prism.

2) $A_{\inf}$: $A = \mathbb{Z}_p[q^{1/p^\infty}]^\wedge$. $\xi = [p]_q$, $\phi(q) = q^p$.
Have to check: ξ is primitive. Check after $q \to 1$.
Note: ϕ is uniquely determined. Something about compatible pth roots and the action of ϕ. It gives $\phi(-) = -^p$.

3) q-dR. $A = \mathbb{Z}_p[[q-1]]$, $\phi : q \mapsto q^p$, $\xi = [p]_q$.

4) Breuil-Kisin cohomology.
Then work over any local field, unlike Wach modules.

$K/\mathbb{Q}_p$ fin. ext.

$\pi \in \mathcal{O}_K$ uniformizer.

$W = W(\mathcal{O}_K/m)$.

$A = W[[u]] \circlearrowleft \phi(u) = u^p$ $\quad \xi = E(u)$

Eisenstein poly of π.

$u \mapsto (\pi)$.

Mod out, get $\mathcal{O}_K$.

<u>Another Ex.</u> R perfectoid ring. $R^\flat$ tilt.

$$R = A_{\inf}(R)/(\xi).$$

$(A_{\inf}, (\xi))$ a perfect prism.

<u>Rem.</u> Perfect prisms give you diamonds.

<u>Rem.</u> Prisms are a kind of non-perfect perfectoid rings.

<u>Rem.</u> Why primitive? They cannot be divided further.

<u>Lemma.</u> A a δ-ring. $f, g \in A$. Assume: g primitive and $f \mid g$. Assume $(f, p) \subseteq$ Jacobson radical. Then, $f = g \cdot \text{unit}$ and f is primitive.

Always completing along the fiber $A/(\xi)$ of the prism.

<u>proof.</u> $g = hf$.

$$\underset{\text{Unit}}{\delta(g)} = h^p \delta(f) + \underbrace{\delta(h) f^p + p\, \delta(h)\delta(f)}_{\text{In Jacobson rad.}}$$

So, $h^p \delta(f)$, h, $\delta(f)$ units, as desired.

IV) Crystalline cohomology via prisms.

Setup. A = p-torsion free δ-ring.

$(A, (p))$ prism.

$\bar{A} = A/(p)$.

R [redacted] over $\bar{A}$.

$\left(\text{Ex: } A = \mathbb{Z}_p, \bar{A} = \mathbb{F}_p. \right)$

<u>Def</u> (Prismatic site). The category

$$(R/A)_{\Delta}$$

with objects

$$\begin{tikzcd} A \arrow[rr, "\delta\text{-ring map}"] \arrow[d] & & B \arrow[d] \\ A/(p) \arrow[r] & R \arrow[r] & B/(p) \end{tikzcd}$$

Some δ-ring.

Denote this by $(R \to \bar{B} \leftarrow B)$.

Send $(R \to \bar{B} \leftarrow B)$ to B. This is the structure sheaf $\mathcal{O}_{\Delta}$.

ϕ Everywhere.

Now, $\Delta_{R/A} \simeq R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$.

$\bar{\Delta}_{R/A} \simeq R\Gamma((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta})$.

ϕ Everywhere. Specific to $\xi = p$.

$$\bar{\mathcal{O}}_{\Delta} = \mathcal{O}_{\Delta}/(p)$$

<u>Rem</u>. No topology for some reason.

<u>Rem</u> 1) $\exists$ a non-affine variant.

2) $\bar{\Delta}_{R/A}$ is R-linear.

<u>Thm A</u>. Assume $R/\bar{A}$ is smooth.

1) Crys. comp: $\phi_A^* \Delta_{R/A} \simeq R\Gamma_{crys}(R/A)$.

2) Cartier: there is a can. iso $\Omega^i_{R/\bar{A}} \simeq H^i(\bar{\Delta}_{R/A})$.

Induced by Bockstein.

Cor. If $X/\bar{A}$ is smooth, then

$$Rf_* \Omega^\bullet_{X/\bar{A}}$$

has a canonical Frobenius descent. Here, $f : X \to \mathrm{Spec}(\bar{A})$.

This says that $R\Gamma_{dR}(X/\bar{A})$ itself descends further.

But, not Gauss-Manin.

Follows also from Ogus-Vologodsky (for ind. cohomology groups): Things w/ int. connection and Higgs bundles.

Key steps of proof.

1) Produce a comp. map $\Delta_{R/A} \longrightarrow \underbrace{\phi_* R\Gamma_{crys}(R/A)}$.

Have to explain how to compute this via $B_\bullet$.

FACT: $R\Gamma_{crys}(R/A)$ is computed by a diagram of pd-thickenings $D \twoheadrightarrow R$ where D is a δ-A-alg.

$$\begin{tikzcd} D \arrow[r, "\phi"] \arrow[d, "pd"] & \phi_* D \arrow[d] \\ R \arrow[r] \arrow[ur, dashed] & (\phi_* D)/p \end{tikzcd}$$

Object of the Prismatic Site.

Formal nonsense $\Delta_{R/A} \longrightarrow \phi_* D_\bullet \simeq \phi_* R\Gamma_{crys}(R/A)$.

2) Relate pd-structures to δ-rings.

Lemma. If B is a p-torsion free δ-ring and $f \in B$ is s.t. $\frac{f^p}{p} \in B$, then $\frac{f^n}{n!} \in B$ $\forall n \geq 2$.

Cor. B = p-torsion free δ-ring.

$f \in B$ s.t. $\bar{f}$ is a non-zero divisor mod (p).

Then,
$$D_B((f)) \cong B\left\{\frac{f^p}{p}\right\} \cong B\left\{\frac{\phi(f)}{p}\right\}$$

V) Prismatic cohomology.

Setup. (A, ξ) prism, $\bar{A} = A/(\xi)$.

Assume (p, ξ) form a reg. sequence. (Irrelevant if you derive.)
$R/\bar{A}$ formally smooth

<u>Main ex.</u>
- $A = \mathbb{Z}_p[q^{1/p^\infty}]$, $\xi = [p]_q$.
- $A = \mathbb{Z}_p[[q-1]]$, $\xi = [p]_q$.
- Breuil-Kisin...

<u>Def</u> (Prismatic site).

$$(R/A)_{\Delta} = \left\{ \begin{tikzcd} A \arrow[rr, "\delta\text{-ring map}"] \arrow[d] & & B \arrow[d] \\ \bar{A} \arrow[r] & R \arrow[r] & B/(\xi) =: \bar{B} \end{tikzcd} \right\}$$

$$\left.\begin{array}{l} \Delta_{R/A} \\ \bar{\Delta}_{R/A} \end{array}\right\} \text{ As before.}$$

$\bar{\Delta}_{R/A} \cong \Delta_{R/A}/(\xi)$, R-linear.

<u>Thm B.</u> 1) $(\phi^* \Delta_{R/A})/\xi \simeq R\Gamma_{dR}(R/\bar{A})$.

Typically cannot move ξ inside.

2) Cartier iso:

$$\Omega^i_{R/\bar{A}} \simeq H^i(\bar{\Delta}_{R/A}) \otimes_A \frac{(\xi^i)}{(\xi)^{i+1}}.$$

(Didn't need this above as $\xi = p$, canonical.)

<u>Key idea</u>. Want to deform to previous situation.

Set $D := A\left\{\frac{\phi(\xi)}{p}\right\} \cong A\left[\frac{\xi^n}{n!}\right]$.

$$\begin{tikzcd}
A \arrow[r, "\phi\circ can"] \arrow[d] & D \arrow[d] \\
\bar{A} \cong A/(\xi) \arrow[r] & D/\phi(\xi) \cong D/(p)
\end{tikzcd}$$

$\xi \longmapsto \phi(\xi)$, divisible by p.

Lem. on prism. elts.

$$\phi(\xi) = \xi^p + p\delta(\xi) = \Big(\underbrace{\frac{\xi^p}{p}}_{\in \text{ Jac. radical}} + \underbrace{\delta(\xi)}_{\text{unit}}\Big)\cdot p$$

We've gone from $(A, (\xi))$ to a crystalline prism. So, base change along $\phi: A \to D$ to reduce Thm B to Thm A.

Thm C. If $(A, \xi) = (A_{inf}(\mathbb{Z}_p^{cycl}), [p]_q)$,

then $\Delta_{R/A} \simeq A\Omega_R$.

as in BMS.

Or, via THH.

Point: $A\Omega_R$ is computed as a cosimplicial δ-ring.

VI) q-dR cohomology. Twisted version of prismatic site.

Setup: $A = \mathbb{Z}_p[[q-1]] \circlearrowright \phi$

$R/\mathbb{Z}_p$ formally smooth.

Goal: q-analogue of crystalline site.

Need $\frac{x^n}{n!}$ in q-world.

Fix a δ-A-algebra D. An ideal $I \subseteq D$. D has q-divided powers along I if

$$\phi(I) \subseteq [p]_q \cdot D.$$

Ex. 1) If $q=1$, same as divided powers. I.e., over $\mathbb{Z}_p[[q-1]]/(q-1)$. Use observation about divided powers in δ-rings.

2) $A \xrightarrow{q \mapsto 1} \mathbb{Z}_p$ is a q-pd-thickening. (A has q-divided powers along kernel.)

$\curvearrowright \phi$

3) $B = A[x,y]$, $x \longmapsto x^p$, $y \longmapsto y^p$.

How to adjoin q-pd of $x-y$. Answer:

$$D := B\left\{ \frac{\phi(x-y)}{[p]_q} \right\}.$$

Not obvious that this is q-pd.

Nat. map ϕ

$$D \longrightarrow \frac{\mathbb{Z}_p[x,y]}{(x-y)}.$$

And, D has q-dp along the kernel.
q-deformed pd-envelope of $\frac{\mathbb{Z}_p[x,y]}{(x-y)}$.

Also, D is free over $A[x]$ on

$$\gamma_{k,p}(x-y) := \frac{(x-y)(x-qy)\cdots(x-q^{k-1}y)}{[k]_q!}$$

Pridham.

$q \to 1$ gives ordinary divided powers.

Def. $(R/A)_{q\text{-crys}} = \left\{ \begin{array}{c} D \longrightarrow R \\ q\text{-pd-thickenings} \end{array} \right\}$

$q\Omega_R$ = derived global sections
of $(D \longrightarrow R) \longmapsto D$.

Thm D. 1) If we choose coordinates on R,

then $q\Omega_R \simeq q\Omega^{*}_{R,\square}$. This proves

Scholze's conjecture.

2) $$H^i\left((q\Omega_R)/[p]_q \right) \simeq \Omega^i_{R/\mathbb{Z}_p} \otimes \mathbb{Z}_p[\zeta_p].$$

Some kind of Cartier.

3) If $R^{(1)} := R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p]$, $\quad \mathbb{Z}_p[\zeta_p] \cong \mathbb{Z}_p[q-1]/[p]_q$

then $$q\Omega_R \simeq \Delta_{R^{(1)}/A}.$$

R doesn't have an associated prism. ∎

So, this doesn't give us a q-theory associated to any prism theory. As in Breuil-Kisin.